Ternary Continued Fractions and the Evenly-Tempered Musical Scale

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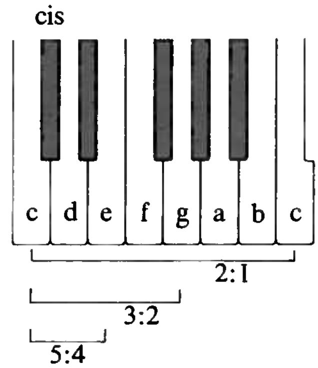
In the evenly-tempered 12-note chromatic scale of western music, two important intervals are well-approximated - the pure fifth with ratio 3/2, and the major third with ratio 5/4. After taking log-ratios, the musical scale can be viewed as an example of the approximation of two irrational numbers by a pair of rational numbers with the same denominator (in this case, 12). A general approach to such problems is provided by the theory of ternary continued fractions.

1. Introduction

The most important intervals in music are the octave, pure fifth, and major third. Perhaps they are consonant to the ear because they are based on simple whole number ratios (2:1, 3:2, and 5:4, respectively). These intervals arise naturally out of the overtone series for a vibrating string. A string vibrating at a frequency f also vibrates at $2f$, $3f$, $4f$, etc. The ratios between the overtones include these basic intervals. There are more complicated intervals than the major third, but they are not heard by the ear strongly enough to be a major factor in tuning.

It is impossible to tune a scale so that all of these intervals come out exact for all the notes. In every system of tuning the ratio of two notes an octave apart is always taken to be exactly 2:1. The attempt of most systems of tuning has been to approximate the fifth and major third, although the accuracy of the fifth often predominates.



For example, the Pythagorean system of tuning is based only on the octave and fifth. All the fifths but one have a ratio of 3:2. It works well for unison melodies and simple melodies with fourths and fifths, but for more complicated melodies with thirds and sixths it sounds dissonant. Furthermore, because some half-steps have different ratios than others, some keys sound more consonant than others. A piece played in *C* will sound more in tune than a piece in *F*#.

The problems of different tunings in different keys can be avoided by restricting attention to tunings in which all the half-steps have the same ratio. Such a scale is called evenly-tempered. The modern day piano and many other instruments are tuned to the 12-note evenly-tempered scale. The basic interval is the half-step.

More generally, we might consider an evenly-tempered scale of n notes in which each unit interval has a ratio $2^{1/n}:1$. If there are k notes in the approximation to the pure fifth, then this interval has a ratio of $2^{k/n}:1$. This will be an irrational number and so can never be exactly 3:2. Similarly for the major third. Thus, in a good evenly-tempered n-note scale there will be numbers h and k such that $2^{h/n} \approx 5/4$ and $2^{k/n} \approx 3/2$. That is, $h/n \approx \log_2 5/4$ and $k/n \approx \log_2 3/2$.

In fact, logarithms make intuitive sense in dealing with intervals. The ear hears a half-step above a half-step as a whole step, so it seems more natural to add the logarithms of the ratios than to multiply the ratios. If one divides the octave into 1200 logarithmic units known as cents, each half-step of the 12-note scale is 100 cents. A true major third has a value of $1200(\log_2 5/4) \approx 386$ cents and a true pure fifth has a value of $1200(\log_2 3/2) \approx 702$ cents. Thus in the 12-note scale the major third is 14 cents sharper than a true major third and the fifth is 2 cents flat.

The object of this paper is to find evenly-tempered scales which give good approximations to the true major third and pure fifth. We are looking for a sequence of pairs of rational numbers with the same denominator, which approximate $\log_2 5/4$ and $\log_2 3/2$. The process by which this is done is known as the theory of ternary continued fractions, and it forms an extension of the idea of ordinary continued fractions. Ordinary continued fractions are used to approximate one number; ternary continued fractions are used to approximate two numbers. They were developed by JACOBI [6]. We will discuss Jacobi's expansion in Section 3 and then an alternative expansion known as the reversed expansion in Section 4. These two expansions can be combined (Section 5) in a way to give us a sequence of approximations which include many of the scales proposed by musical theorists (Section 6).

The main ideas of this paper are due to BARBOUR ([2], and [3], Chapter 6), who gives further details about the musical implications of these results. Here we shall concentrate in more detail on the mathematical development of ternary continued fractions.

2. Ordinary continued fractions

The theory of ordinary continued fractions provides a powerful method of finding a sequence of rational approximations to an irrational number; see, for example HARDY and WRIGHT ([5], Chapter 10) or BAKER ([1], Chapter 6).

For our purposes we shall restrict attention to the case where $\alpha_0 < 1$ is an irrational positive number. The continued fraction expansion for α_0 can be developed as follows. Define a sequence of integers $p_i \geqslant 1$ and positive real numbers $\alpha_i < 1$ by

$$p_i = [\alpha_i^{-1}], \quad \alpha_{i+1} = \alpha_i^{-1} - p_i, \quad i \geqslant 0,$$

where [.] denotes integer part. At each stage we approximate a remainder by the integer part of its reciprocal. Thus,

$$\alpha_i = [\alpha_{i+1} + p_i]^{-1}. \tag{2.1}$$

If for some j we approximate $\alpha_{j+1} \approx 0$ and use backwards recursion in (2.1), then we obtain a rational approximation for α_0 which can be written as a ratio of integers, A_j/B_j say.

For more formal mathematical work it is convenient to set out this procedure in terms of linear transformations. Set $U_0 = \alpha_0$, $V_0 = 1$, and define sequences of positive real numbers by

$$\begin{bmatrix} U_{i+1} \\ V_{i+1} \end{bmatrix} = \begin{bmatrix} -p_i & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U_i \\ V_i \end{bmatrix}$$

where $p_i = [V_i/U_i]$. Thus V_i/U_i is the same as α_i^{-1} above. Next define sequences of integers A_i, B_i by

$$\begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} A_{i-2} & A_{i-1} \\ B_{i-2} & B_{i-1} \end{bmatrix} \begin{bmatrix} 1 \\ p_i \end{bmatrix}$$

with initial conditions

$$\begin{bmatrix} A_{-2} & A_{-1} \\ B_{-2} & B_{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A_i/B_i is called the *i-th convergent* for α_0 and $A_i/B_i \to \alpha_0$ as $i \to \infty$. This is classically expressed by the equality

$$\alpha_0 = \cfrac{1}{p_0 + \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cdots}}}.$$

Further it can be shown that the speed of convergence is quite rapid,

$$| A_i/B_i - \alpha_0 | < 1/B_i^2. \tag{2.2}$$

The multivariate version of Dirichlet's theorem (see for example HARDY and WRIGHT [5], pp. 169-170, or BAKER [1], pp. 56-59) says that, given positive numbers $\beta_1, \ldots, \beta_m$ and an integer $Q^* > 0$, there exist integers $Q \leqslant Q^*$ and

$P_1, \ldots, P_m$ such that

$$| P_j/Q - \beta_j | < Q^{-1-1/m}, \quad j = 1, \ldots, m .$$

From (2.2) we see that ordinary continued fractions ($m = 1$) always achieve this inequality. Unfortunately the approximations from ternary continued expansions ($m = 2$) are not so powerful in general.

3. Jacobi's ternary continued fraction expansion

Let $U_0 < V_0 < W_0 = 1$ be three positive numbers. The objective is to find integers (A_i, B_i, C_i) such that A_i: B_i: C_i approximates U_0: V_0: W_0. To ensure the expansion is well defined, suppose U_0, V_0, W_0 are linearly independent over the rationals.

First define sequences U_i, V_i, W_i by the following recurrence formulae for $i \geqslant 0$,

$$\begin{bmatrix} U_{i+1} \\ V_{i+1} \\ W_{i+1} \end{bmatrix} = \begin{bmatrix} -p_i & 1 & 0 \\ -q_i & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} \tag{3.1}$$

where

$$p_i = [V_i/U_i] \ , \ q_i = [W_i/U_i] .$$

Next define 3 sequences of integers recursively for $i \geqslant 0$ by

$$\begin{bmatrix} A_i \\ B_i \\ C_i \end{bmatrix} = \begin{bmatrix} A_{i-3} & A_{i-2} & A_{i-1} \\ B_{i-3} & B_{i-2} & B_{i-1} \\ C_{i-3} & C_{i-2} & C_{i-1} \end{bmatrix} \begin{bmatrix} 1 \\ p_i \\ q_i \end{bmatrix} \tag{3.2}$$

with initial conditions

$$\begin{bmatrix} A_{-3} & A_{-2} & A_{-1} \\ B_{-3} & B_{-2} & B_{-1} \\ C_{-3} & C_{-2} & C_{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

This is the Jacobi expansion for (U_0, V_0, W_0) (Jacobi [6]); see also Daus [4]. The triple (A_i, B_i, C_i) is known as the *i-th convergent set*, the pair (q_i, p_i) is the *i-th partial quotient set*, and the triple (U_i, V_i, W_i) is the *i-th complete quotient set*. Thus, the approximation of U_0: V_0: W_0 can be depicted as

$$1 : p_0 + \cfrac{1}{q_1 + \cfrac{p_2 + \cfrac{1}{q_3 + \cdots}}{q_2 + \cfrac{p_3 + \cdots}{q_3 + \cdots}}} : q_0 + \cfrac{p_1 + \cfrac{1}{q_2 + \cfrac{p_3 + \cdots}{q_3 + \cdots}}}{q_1 + \cfrac{p_2 + \cfrac{1}{q_3 + \cdots}}{q_2 + \cfrac{p_3 + \cdots}{q_3 + \cdots}}} .$$

The 3×3-matrix in (3.1) has determinant $+1$, so that $(U_{i+1}, V_{i+1}, W_{i+1})$ will be linearly independent when (U_i, V_i, W_i) are. In particular $U_{i+1}\neq 0$ which is the only requirement that needs to be satisfied in order to continue the expansion for another step. Thus, the expansion can be continued infinitely when (U_0, V_0, W_0) are linearly independent.

From the definition of p_i and q_i we see that $U_{i+1}<U_i$ and $V_{i+1}<U_i$. Thus, since $W_{i+1} = U_i$, we get

$$U_{i+1}<W_{i+1}\ ,\ V_{i+1}<W_{i+1}\ .$$

Therefore, at each step $0\leqslant p_i\leqslant q_i$ and $q_i\geqslant 1$.

Let $\sigma_{1,i}=V_i/U_i$, $\sigma_{2,i}=W_i/U_i$. Then $\sigma_{1,i}<\sigma_{2,i}$ and $\sigma_{2,i}>1$ at each step. Note that $\sigma_{1,i}$ and $\sigma_{2,i}$ play a role similar to that of α_i^{-1} in Section 2. Inverting the matrix in (3.1) we see that

$$\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & p_i \\ 0 & 1 & q_i \end{bmatrix} \begin{bmatrix} U_{i+1} \\ V_{i+1} \\ W_{i+1} \end{bmatrix} \tag{3.3}$$

Thus

$$\sigma_{1,i} = p_i + \frac{1}{\sigma_{2,i+1}}\ ,\quad \sigma_{2,i} = q_i + \frac{\sigma_{1,i+1}}{\sigma_{2,i+1}}\ ,$$

where

$$1/\sigma_{2,i+1}<1 \text{ and } \sigma_{1,i+1}/\sigma_{2,i+1}<1\ .$$

Also, if $p_i=q_i$, then, since $\sigma_{1,i}<\sigma_{2,i}$, we must have $\sigma_{1,i+1}>1$, and so $p_{i+1}\geqslant 1$.

It can be shown (PERRON [7], [8]) that these properties of the partial quotient sets (that is, that $0\leqslant p_i\leqslant q_i$ and $q_i\geqslant 1$ for all i, and that $p_i=q_i\Rightarrow p_{i+1}\geqslant 1$) guarantee that the expansion is unique.

The first convergent sets are

$$\begin{aligned} A_0 &= 1 & B_0 &= p_0 & C_0 &= q_0 \\ A_1 &= q_1 & B_1 &= q_1p_0+1 & C_1 &= q_1q_0+p_1\ . \end{aligned}$$

Thus

$$A_1>0,\quad B_1>0,\quad C_1>0\ .$$

Let S_i stand for either A_i, B_i, or C_i. Then (3.2) together with the facts that $q_i\geqslant 1$ and $S_i>0$ for $i\geqslant 1$ show that $S_i>S_{i-1}$ for $i\geqslant 4$. Therefore $\{S_i\}_{i\geqslant 3}$ forms a strictly increasing sequence (DAUS [4], p. 281).

The following identity is easily proved by induction for $i\geqslant 0$ using (3.3):

$$\begin{bmatrix} U_0 \\ V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} A_{i-3} & A_{i-2} & A_{i-1} \\ B_{i-3} & B_{i-2} & B_{i-1} \\ C_{i-3} & C_{i-2} & C_{i-1} \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}. \tag{3.4}$$

Dividing two rows in (3.4) by U_i and taking their ratio gives

$$\frac{U_0}{W_0} = \frac{A_{i-3}+\sigma_{1,i}\,A_{i-2}+\sigma_{2,i}\,A_{i-1}}{C_{i-3}+\sigma_{1,i}\,C_{i-2}+\sigma_{2,i}\,C_{i-1}} \tag{3.5}$$

If we approximate $\sigma_{1,i}$ by p_i and $\sigma_{2,i}$ by q_i we might hope that we would get an approximation to U_0/W_0. That is, we would hope,

$$\frac{A_{i-3}+p_i\,A_{i-2}+q_i\,A_{i-1}}{C_{i-3}+p_i\,C_{i-2}+q_i\,C_{i-1}} = \frac{A_i}{C_i} \approx \frac{U_0}{W_0}\,.$$

In fact A_i/C_i and B_i/C_i do converge to U_0/W_0 and V_0/W_0 respectively, but convergence is not nearly so swift as it is for ordinary continued fractions. We shall show that

$$|\,A_i/C_i - U_0/W_0\,| + |\,B_i/C_i - V_0/W_0\,| = O(C_i^{-1})$$

in contrast to (2.2) for ordinary continued fractions.

THEOREM 1. (VAISÄLÄ [9]). *Let $U_0 < V_0 < W_0$ be positive numbers linearly independent over the rationals. Let $(A_i,\, B_i,\, C_i)$ be the i-th convergent set in the Jacobi expansion. Then $A_i/C_i \to U_0/W_0$ and $B_i/C_i \to V_0/W_0$ as $i \to \infty$.*

PROOF. Set $H_i = A_i - (U_0/W_0)C_i$ and $K_i = B_i - (V_0/W_0)C_i$. Then

$$A_i/C_i - U_0/W_0 = H_i/C_i\ ,\quad B_i/C_i - V_0/W_0 = K_i/C_i\ .$$

Now $C_i \uparrow \infty$ so if there is an upper bound on H_i and K_i, convergence is guaranteed.

From (3.5) we see that

$$H_{i-3}+\sigma_{1,i}\,H_{i-2}+\sigma_{2,i}\,H_{i-1} = 0\,.$$

Therefore,

$$H_{i-1} = -1/\sigma_{2,i}\,[H_{i-3}+\sigma_{1,i}\,H_{i-2}]\,.$$

Replacing i by $i+1$ we get

$$H_i = -1/\sigma_{2,i+1}\,[H_{i-2}+\sigma_{1,i+1}\,H_{i-1}]\,. \tag{3.6}$$

Also, from (3.2) we have

$$H_i = H_{i-3}+p_i\,H_{i-2}+q_i\,H_{i-1}\,. \tag{3.7}$$

There are two cases to consider. If H_{i-2}, H_{i-1} have the same sign, then by (3.6) H_i has the opposite sign, so from (3.7)

$$|\,H_i\,| = |\,H_{i-3}\,| - p_i\,|\,H_{i-2}\,| - q_i\,|\,H_{i-1}\,| < |\,H_{i-3}\,|\,.$$

If H_{i-2}, H_{i-1} have the opposite signs, then from (3.6)

$$\begin{aligned}|\,H_i\,| &< \max\,\{\sigma_{2,i+1}^{-1}\,|\,H_{i-2}\,|\,,\,(\sigma_{1,i+1}/\sigma_{2,i+1})\,|\,H_{i-1}\,|\,\}\\ &< \max\,\{\,|\,H_{i-2}\,|\,,\,|\,H_{i-1}\,|\,\}.\end{aligned}$$

Therefore in either case $|H_i| < \max\{|H_{i-3}|, |H_{i-2}|, |H_{i-1}|\}$.

Thus $|H_i|$ is bounded above by $\max\{|H_0|, |H_1|, |H_2|\}$. Similarly reasoning puts an upper bound on $|K_i|$. Hence the theorem follows. □

In the theory of ordinary continued fractions, we know that any infinite sequence of positive integers is the continued fraction expansion of some positive irrational number. We have an analogous result for ternary continued fractions. Let $\{(p_i, q_i)\}_i$ be a sequence of pairs of integers such that $0 \leqslant p_i \leqslant q_i$, $q_i \geqslant 1$ for all i, and if $p_i = q_i$ then $p_{i+1} \geqslant 1$. Then $\{(p_i, q_i)\}_i$ is the Jacobi expansion for some pair of positive numbers α_0 and β_0 and $\{\alpha_0, \beta_0, 1\}$ are linearly independent over the rationals. For further details see PERRON [7], [8].

The methods of this section can be extended in a straightforward way to give a sequence of simultaneous rational approximations to more than two irrational numbers; see PERRON [7].

4. THE REVERSED EXPANSION

There is another expansion due to BARBOUR [2] that can be used to get approximations to $U_0 : V_0 : W_0$. In the Jacobi expansion one always divided by U_i at the i-th step. In the following, the reversed expansion, one divides by V_i. It is defined by the following recursion formulae in matrix form,

$$\begin{pmatrix} U_{i+1} \\ V_{i+1} \\ W_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & -p_i & 0 \\ 0 & -q_i & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} U_i \\ V_i \\ W_i \end{pmatrix} \tag{4.1}$$

where

$$p_i = [U_i / V_i] \text{ and } q_i = [W_i / V_i].$$

Set

$$\begin{pmatrix} A_i \\ B_i \\ C_i \end{pmatrix} = \begin{pmatrix} 1 & A_{i-2} & A_{i-1} \\ 0 & B_{i-2} & B_{i-1} \\ 0 & C_{i-2} & C_{i-1} \end{pmatrix} \begin{pmatrix} p_i \\ 1 \\ q_i \end{pmatrix} \tag{4.2}$$

with the same initial conditions as the Jacobi expansion (cf. (3.2)). Schematically, we can write for this approximation of $U_0 : V_0 : W_0$

$$p_0 + \cfrac{p_1 + \cfrac{p_2 + \cfrac{p_3 + \cdots}{q_3 + \cdots}}{q_2 + \cfrac{1}{q_3 + \cdots}}}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdots}}} : 1 : q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cdots}}}.$$

It is interesting to note that $\{q_i\}_{i \geq 0}$ is just the ordinary continued fraction expansion for V_0/W_0. Also, the B_i/C_i are just the ordinary convergents. Thus the reversed expansion is not as symmetrical as the Jacobi expansion. □

Since the 3×3-matrix in (4.1) has determinant 1, we have again that, if U_i, V_i, W_i are linearly independent over the rationals, then U_{i+1}, V_{i+1}, W_{i+1} will be, and in particular $V_{i+1} \neq 0$. Thus, if U_0, V_0, W_0 are linearly independent, we can define an infinite reversed expansion for them.

In matrix form we have the following identities for $i \geq 0$,

$$\begin{bmatrix} U_0 \\ V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} 1 & A_{i-2} & A_{i-1} \\ 0 & B_{i-2} & B_{i-1} \\ 0 & C_{i-2} & C_{i-1} \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix}$$

which follow easily from (4.1) by induction as

$$\begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & p_i \\ 0 & 0 & 1 \\ 0 & 1 & q_i \end{bmatrix} \begin{bmatrix} U_{i+1} \\ V_{i+1} \\ W_{i+1} \end{bmatrix} \tag{4.3}$$

Let

$$\tau_{1,i} = U_i/V_i \quad \tau_{2,i} = W_i/V_i .$$

Then the recursion formulae become

$$\tau_{1,i} = p_i + \tau_{1,i+1}/\tau_{2,i+1} \quad \tau_{2,i} = q_i + 1/\tau_{2,i+1}$$

where

$$\tau_{1,i+1}/\tau_{2,i+1} < 1 \text{ and } 1/\tau_{2,i+1} < 1 .$$

As in the Jacobi expansion we have convergence of A_i: B_i: C_i to U_0: V_0: W_0, but the proof will be postponed until the next section where a more general theorem is proved.

5. The mixed expansion

In the problem of finding evenly-tempered scales one is interested in scales with small numbers of notes. In both the Jacobi and reversed expansions C_i increases too rapidly to give many interesting scales; see Tables 1(a) and 1(b). So the slow mixed expansion was devised by Barbour [2] to slow the growth of C_i. At each step of the slow mixed expansion, one divides by U_i or V_i whichever is larger.

TABLE 1. Ternary continued expansions for ($\log_2 5/4$, $\log_2 3/2$, 1), adapted from BARBOUR [2].

(a) Jacobi Expansion

p_i	q_i	A_i	B_i	C_i	Error A_i	Error B_i	Total Error	H_{i+3}	K_{i+3}
1	3	1	1	3	14	−302	316	0.03	−0.75
0	1	1	2	3	14	98	112	0.03	0.24
1	7	8	15	25	−2	18	20	−0.04	0.37
0	1	9	16	28	−1	−16	17	−0.01	−0.38
0	1	10	18	31	0.8	−5.2	6.0	0.02	−0.13
0	2	28	51	87	−0.1	1.4	1.5	−0.008	0.1

(b) Reversed Expansion

p_i	q_i	A_i	B_i	C_i	Error A_i	Error B_i	Total Error	H_{i+3}	K_{i+3}
0	1	0	1	1	−386	498	884	−0.32	0.42
0	1	0	1	2	−386	−102	488	−0.64	−0.17
1	2	1	3	5	−146	18	164	−0.61	−0.08
2	2	4	7	12	14	−2	16	0.14	−0.02
0	3	13	24	41	−6	1	7	−0.20	0.02
0	1	17	31	53	−1.4	−0.1	1.5	−0.06	−0.003

(c) Slow Mixed Expansion

Step	p_i	q_i	A_i	B_i	C_i	Error A_i	Error B_i	Total Error	H_{i+3}	K_{i+3}
R	0	1	0	1	1	−386	498	884	−0.32	0.41
R	0	1	0	1	2	−386	−102	488	−0.64	−0.17
J	0	1	1	1	2	214	−102	316	0.36	−0.17
J	0	1	1	2	3	14	98	112	0.03	0.25
R	0	1	2	3	5	94	18	112	0.39	0.08
J	0	1	2	4	7	−43	−16	59	−0.25	−0.09
R	0	1	4	7	12	14	−2	16	0.14	−0.02
R	0	1	6	11	19	−7	−7	14	−0.12	−0.11
R	0	1	10	18	31	0.8	−5.2	6.0	0.02	−0.13
J	0	1	11	20	34	1.9	3.9	5.8	0.05	0.11
J	0	1	17	31	53	1.4	−0.1	1.5	−0.06	−0.003

(d) Other Scales

A_i	B_i	C_i	Error A_i	Error B_i	Total Error
5	10	17	−33	4	37
7	13	22	−4	7	12

In a general mixed expansion, the choice of divisor at each stage can be arbitrary. The formulae for U_{i+1}, V_{i+1}, W_{i+1} are the same as for the Jacobi or the reversed expansion depending on whether one divides by U_i or V_i at step i. However, new formulae for A_i, B_i, C_i are needed.

Let J denote a Jacobi step and R a reversed step. Let $k = k(i)$ denote the number of steps between the present step, i, and the last previous J step. Let $k=i$ if there have been no J steps. Let S_i stand for either A_i, B_i, or C_i. Then define S_i by the following recursion formula,

$$S_i = p_i\, S_{i-k-3} + S_{i-2} + q_i\, S_{i-1} \quad \text{if step } i \text{ is } R \tag{5.1}$$

$$S_i = S_{i-k-3} + p_i\, S_{i-2} + q_i\, S_{i-1} \quad \text{if step } i \text{ is } J\ , \tag{5.2}$$

with the same initial conditions as for the Jacobi expansion. It is still clear that $C_i \to \infty$ as $i \to \infty$. The following identities are useful and can be proved by induction.

$$\begin{bmatrix} U_0 \\ V_0 \\ W_0 \end{bmatrix} = \begin{bmatrix} A_{i-k-3} & A_{i-2} & A_{i-1} \\ B_{i-k-3} & B_{i-2} & B_{i-1} \\ C_{i-k-3} & C_{i-2} & C_{i-1} \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ W_i \end{bmatrix} \tag{5.3}$$

If $i=0$, then $k=0$ and the formulae follow from the initial conditions. Suppose they hold at step i. If step i is J then from (3.3) and (5.2)

$$\begin{aligned} &U_i S_{i-k-3} + V_i S_{i-2} + W_i S_{i-1} = \\ &= \begin{bmatrix} U_{i+1} \\ V_{i+1} \\ W_{i+1} \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & p_i & q_i \end{bmatrix} \begin{bmatrix} S_{i-k-3} \\ S_{i-2} \\ S_{i-1} \end{bmatrix} \\ &= U_{i+1} S_{i-2} + V_{i+1} S_{i-1} + W_{i+1} S_i \end{aligned}$$

which is correct because at the $(i+1)$-th step k becomes 0.

Similarly, if step i is R then from (4.3) and (5.1),

$$\begin{aligned} &U_i S_{i-k-3} + V_i S_{i-2} + W_i S_{i-1} \\ &= (U_{i+1} + p_i W_{i+1}) S_{i-k-3} + W_{i+1} S_{i-2} + (V_{i+1} + q_i W_{i+1}) S_{i-1} \\ &= U_{i+1}\ S_{(i-2)-(k+1)} + V_{i+1}\ S_{i-1} + W_{i+1}\ S_i \end{aligned}$$

which is correct because at the $(i+1)$-th step k becomes $k+1$. Hence (5.3).

Barbour [2] does not discuss the convergence properties of the reversed and slow mixed expansions, but the argument of theorem 1 can be extended to prove the following result.

Theorem 2. *Let $U_0 < V_0 < W_0$ be positive numbers linearly independent over the rationals. Given an arbitrary sequence of Jacobi and reversed steps we can expand them in an infinite mixed expansion. Let A_i, B_i, C_i be defined as above. Then $\lim_{i\to\infty} A_i / C_i = U_0 / W_0$ and $\lim_{i\to\infty} B_i / C_i = V_0 / W_0$.*

Proof. As in the proof for the Jacobi expansion we only need to show that $H_i = A_i - (U_0/W_0) C_i$ and $K_i = B_i - (V_0/W_0) C_i$ are bounded in absolute value. Letting $\sigma_{1,i} = V_i/U_i$, $\sigma_{2,i} = W_i/U_i$, $\tau_{1,i} = U_i/V_i$, $\tau_{2,i} = W_i/V_i$ then we still know that $\sigma_{2,i}>1$, $\sigma_{2,i}/\sigma_{1,i}>1$, $\tau_{2,i}>1$, $\tau_{2,i}/\tau_{1,i}>1$ since $U_i<W_i$ and $V_i<W_i$ for all $i\geqslant 0$.

There are two cases to consider.

1. Suppose step i is J. Then $H_i=H_{i-k-3}+p_iH_{i-2}+q_iH_{i-1}$, and $H_i=-1/\sigma_{2,i+1}[H_{i-2}+\sigma_{1,i+1}H_{i-1}]$ or $H_i=-1/\tau_{2,i+1}[\tau_{1,i+1}H_{i-2}+H_{i-1}]$ depending on whether step $i+1$ is J or R. If H_{i-2}, H_{i-1} have opposite signs, then

$$|H_i| \leqslant \max\{|H_{i-2}|, |H_{i-1}|\}.$$

If H_{i-2}, H_{i-1} have the same sign, then H_i has the opposite sign and $|H_i| \leqslant |H_{i-k-3}|$.

2. Suppose step i is R. Then $H_i=p_iH_{i-k-3}+H_{i-2}+q_iH_{i-1}$, and $H_i=-1/\sigma_{2,i+1}[H_{i-k-3}+\sigma_{1,i+1}H_{i-1}]$ or $H_i=-1/\tau_{2,i+1}[\tau_{1,i+1}H_{i-k-3}+H_{i-1}]$ depending on whether step $i+1$ is J or R. If H_{i-k-3}, H_{i-1} have opposite signs, then

$$|H_i| \leqslant \max\{|H_{i-k-3}|, |H_{i-1}|\}.$$

If H_{i-k-3}, H_{i-1} have the same sign, then H_i has the opposite sign and $|H_i| \leqslant |H_{i-2}|$. Therefore, in any case

$$|H_i| \leqslant \max\{|H_{i-k-3}|, |H_{i-2}|, |H_{i-1}|\}.$$

Thus the $|H_i|$ sequence is bounded. Similar reasoning shows that the $|K_i|$ sequence is bounded and so the theorem follows. □

The proof of the above theorem does not depend on the particular sequence of Jacobi and reversed steps used, so the convergence of the Jacobi, reversed, and slow mixed expansions follow as special cases.

Note that we could reverse the order of U_0 and V_0 in Section 3 without affecting the validity of the Jacobi expansion. The effect would be the same as using a mixed expansion with one R step followed thereafter by J steps under the original order.

The slow mixed expansion of this section has been devised to slow the growth of the denominator C_i. Alternatively we could divide by the smaller of U_i and V_i at each step in order to speed the growth of C_i. We shall not explore this possibility further here.

6. Discussion of musical scales

Tables 1(a), 1(b) and 1(c) give the results of the Jacobi, reversed, and slow mixed expansions applied to the numbers $U_0=\log_2 5/4 \approx 0.3219$, $V_0=\log_2 3/2 \approx 0.5850$, $W_0=1$. Here C_i represents the number of notes in an octave, A_i the number of notes in the major third, and B_i the number of notes in the fifth. In order for A_i and B_i to be the best approximations for the denominator C_i we must have

$$|H_i| = |A_i-(U_0/W_0)C_i| < \tfrac{1}{2} \text{ and } |K_i| = |B_i-(V_0/W_0)C_i| < \tfrac{1}{2}.$$

For all of the interesting cases these inequalities are easily satisfied.

The errors in A_i/C_i and B_i/C_i are measured in cents,

$$\text{Error } A_i = 1200(A_i/C_i - U_0/W_0)$$

$$\text{Error } B_i = 1200(B_i/C_i - V_0/W_0)\,.$$

The total error is taken to be $|$ Error A_i $|$ + $|$ Error B_i $|$.

The scales from these expansions include many of the important scales proposed by musical theorists and several scales in use by various non-western cultures. The following comments are taken from BARBOUR ([2] and [3], Chapter 6) who discusses these and other scales in more detail.

Two of the scales with fewer than 12 notes are worth mentioning. According to Barbour, Javanese music is based on an evenly-tempered 5-note scale, and Siamese music, on an evenly-tempered 7-note scale.

In western 12-note music there are 5 whole steps and 2 (diatonic) half-steps in the octave. If each whole step is split into a diatonic and a chromatic half-step, there are 7 diatonic and 5 chromatic half-steps. In an evenly-tempered tuning the ratio between these two kinds of half-step is taken as 1:1. If instead one takes the ratio to be 2:1 one gets $7\times2+5\times1=19$ notes in the octave. Thus, one can get a non-evenly-tempered 12-note scale by taking 12 notes out of an evenly-tempered 19 scale.

Other important scales in Table 1(c) which can be interpreted in this way are those with 31 notes (ratio 3:2) and 53 notes (ratio 4:5).

Arabian music is based on a 17-note evenly-tempered scale. This scale has a good fifth (within 4 cents), but the major third is very flat being about midway between a true major third and minor third. The poorness of the third probably explains why it does not appear in the expansions.

The 22-note scale is one important scale missing from these tables though it appears under a more general mixed expansion. It is interesting to note that both the 19 and 22-note scales form better approximations in terms of total error than the 25-note scale in the Jacobi expansion. Thus from this point of view, the convergents of the Jacobi expansion are not necessarily best possible approximations. In this respect Jacobi ternary continued fractions are weaker than ordinary continued fractions, because in ordinary continued fractions one gets a best possible approximation at every step.

From the musical point of view, the accuracy of the pure fifth is more important than the accuracy of the major third. The concept of total error does not take this feature into account. The 12-note scale has a better pure fifth than any evenly-tempered scale with fewer than 41 notes.

The only possible systems of multiple division of the octave which could have any practical significance are the 19 and 22-note scales. Any more notes than that would make an instrument extremely unwieldy to play. Further, as it does not seem likely that the 19- or 22-note scales will come into widespread acceptance, most music seems destined to remain in the evenly-tempered 12-note scale.

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